

Weak drifts of infinitely divisible distributions and their applications

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Weak drift of an infinitely divisible distribution μ on \mathbb{R}^d is defined by analogy with weak mean; properties and applications of weak drift are given. When μ has no Gaussian part, the weak drift of μ equals the minus of the weak mean of the inversion μ' of μ . Applying the concepts of having weak drift 0 and of having weak drift 0 absolutely, the ranges, the absolute ranges, and the limit of the ranges of iterations are described for some stochastic integral mappings. For Lévy processes the concepts of weak mean and weak drift are helpful in giving necessary and sufficient conditions for the weak law of large numbers and for the weak version of Shtatland's theorem on the behavior near $t = 0$; those conditions are obtained from each other through inversion. KEY WORDS: Infinitely divisible distribution; weak mean; weak drift; inversion; stochastic integral mapping; weak law of large numbers; Shtatland's theorem.

1. INTRODUCTION

This paper introduces the notion of weak drift of an infinitely divisible distribution and applies it, first, to the relations between inversions of infinitely divisible distributions and conjugates of stochastic integral mappings studied in Sato [11] and, second, to the weak law of large numbers for Lévy processes and the weak version of Shtatland's theorem on the behavior of Lévy processes near $t = 0$.

The basic notions in this paper are as follows. Let $ID = ID(\mathbb{R}^d)$ be the class of infinitely divisible distributions on \mathbb{R}^d . The Lévy–Khintchine triplet $(A_\mu, \nu_\mu, \gamma_\mu)$ of $\mu \in ID$ consisting of the Gaussian covariance matrix A_μ , the Lévy measure ν_μ , and the location parameter γ_μ is given by the formula

$$\hat{\mu}(z) = \exp \left[-\frac{1}{2} \langle z, A_\mu z \rangle + \int_{\mathbb{R}^d} (e^{i \langle z, x \rangle} - 1 - i \langle z, x \rangle 1_{\{|x| \leq 1\}}(x)) \nu_\mu(dx) + i \langle \gamma_\mu, z \rangle \right] \quad (1.1)$$

for the characteristic function $\hat{\mu}(z)$, $z \in \mathbb{R}^d$, of μ . Recall that $\nu_\mu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu_\mu(dx) < \infty$. If $\int_{|x| \leq 1} |x| \nu_\mu(dx) < \infty$, then

$$\hat{\mu}(z) = \exp \left[-\frac{1}{2} \langle z, A_\mu z \rangle + \int_{\mathbb{R}^d} (e^{i \langle z, x \rangle} - 1) \nu_\mu(dx) + i \langle \gamma_\mu^0, z \rangle \right],$$

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where γ_μ^0 is called the drift of μ . If $\int_{|x|>1} |x| \nu_\mu(dx) < \infty$, then

$$\widehat{\mu}(z) = \exp \left[-\frac{1}{2} \langle z, A_\mu z \rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle) \nu_\mu(dx) + i\langle m_\mu, z \rangle \right],$$

where m_μ is the mean of μ . Let $ID_0 = \{\mu \in ID : A_\mu = 0\}$. For $\mu \in ID_0$ the inversion $\mu' \in ID_0$ of μ is defined as $\nu_{\mu'}(B) = \int_{\mathbb{R}^d \setminus \{0\}} 1_B(|x|^{-2}x) |x|^2 \nu_\mu(dx)$ for all B in $\mathcal{B}(\mathbb{R}^d)$, the class of Borel sets in \mathbb{R}^d , and $\gamma_{\mu'} = -\gamma_\mu + \int_{|x|=1} x \nu_\mu(dx)$. Any $\mu \in ID_0$ has its inversion $\mu' \in ID_0$ and we have $\mu'' = \mu$. The inversion μ' has drift $\gamma_{\mu'}^0$ if and only if μ has mean m_μ ; we have $\gamma_{\mu'}^0 = -m_\mu$. Many other properties of the inversion are given in [8, 11]. For example, for $0 < \alpha < 2$, μ is α -stable if and only if μ' is $(2 - \alpha)$ -stable, and μ is strictly α -stable if and only if μ' is strictly $(2 - \alpha)$ -stable.

We use the following notation throughout this paper: $\mathfrak{C}_0 = \mathfrak{C} \cap ID_0$ for any $\mathfrak{C} \subset ID$; $\mathfrak{C}' = \{\mu' : \mu \in \mathfrak{C}\}$ for any $\mathfrak{C} \subset ID_0$.

The notions that a distribution $\mu \in ID$ has weak mean m_μ and that $\mu \in ID$ has weak mean m_μ absolutely are introduced in [9]. In Section 2 of this paper we will recall those definitions and then define the notions that a distribution $\mu \in ID$ has weak drift γ_μ^0 and that $\mu \in ID$ has weak drift γ_μ^0 absolutely. Properties of weak means and weak drifts are in parallel. Moreover we will prove that μ' has weak drift $\gamma_{\mu'}^0$ if and only if μ has weak mean m_μ and that $\gamma_{\mu'}^0 = -m_\mu$.

Let $\{X_t^{(\rho)} : t \geq 0\}$ be a Lévy process on \mathbb{R}^d such that $\mathcal{L}(X_1^{(\rho)})$, the distribution of $X_1^{(\rho)}$, equals ρ . We consider improper stochastic integrals with respect to $\{X_t^{(\rho)}\}$ in two cases.

- (1) Let $0 < c \leq \infty$ and let $f(s)$ be a locally square-integrable function on $[0, c)$. We say that the improper stochastic integral $\int_0^{c-} f(s) dX_s^{(\rho)}$ is definable if $\int_0^q f(s) dX_s^{(\rho)}$ is convergent in probability as $q \uparrow c$. Define the mapping Φ_f from ρ to $\Phi_f \rho = \mathcal{L}(\int_0^{c-} f(s) dX_s^{(\rho)})$; its domain $\mathfrak{D}(\Phi_f)$ is the class of $\rho \in ID$ such that $\int_0^{c-} f(s) dX_s^{(\rho)}$ is definable.
- (2) Let $0 < c < \infty$ and let $f(s)$ be a locally square-integrable function on $(0, c]$. We say that the improper stochastic integral $\int_{0+}^c f(s) dX_s^{(\rho)}$ is definable if $\int_p^c f(s) dX_s^{(\rho)}$ is convergent in probability as $p \downarrow 0$. Define the mapping Φ_f from ρ to $\Phi_f \rho = \mathcal{L}(\int_{0+}^c f(s) dX_s^{(\rho)})$; its domain $\mathfrak{D}(\Phi_f)$ is the class of $\rho \in ID$ such that $\int_{0+}^c f(s) dX_s^{(\rho)}$ is definable.

In any of the cases (1) and (2), Φ_f is called a stochastic integral mapping. Its range $\mathfrak{R}(\Phi_f) = \{\Phi_f \rho : \rho \in \mathfrak{D}(\Phi_f)\}$ is a subclass of ID . If $c < \infty$ and $\int_0^c f(s)^2 ds < \infty$, then $\int_0^{c-} f(s) dX_s^{(\rho)} = \int_{0+}^c f(s) dX_s^{(\rho)} = \int_0^c f(s) dX_s^{(\rho)}$ for all $\rho \in ID$. If $f = f_h$ is defined from a function h in some way (see Section 3 for precise formulation), the stochastic integral mapping Φ_f is denoted by Λ_h as in [11]. By the transformation of $h(u)$ to $h^*(u) = h(u^{-1})u^{-4}$ the conjugate of Λ_h is defined by Λ_{h^*} and denoted by $(\Lambda_h)^*$ or

Λ_h^* . Thus $\Lambda_h^* = \Lambda_{h^*} = \Phi_{f_h^*}$ and $(\Lambda_h^*)^* = \Lambda_h$. The relations of $\mathfrak{D}(\Lambda_h)$ and $\mathfrak{R}(\Lambda_h)$ with $\mathfrak{D}(\Lambda_h^*)$ and $\mathfrak{R}(\Lambda_h^*)$ are studied in [11]. It is closely connected with the inversion. Thus $\rho \in \mathfrak{D}(\Lambda_h)_0$ and $\Lambda_h \rho = \mu$ if and only if $\rho' \in \mathfrak{D}(\Lambda_h^*)_0$ and $\Lambda_h^* \rho' = \mu'$. In the description of $\mathfrak{R}(\Lambda_h)$ and the range $\mathfrak{R}^0(\Lambda_h)$ of absolutely definable Λ_h (see Section 3 for definition), the conditions of having weak mean 0 and of having weak mean 0 absolutely are sometimes useful, as is shown in [9, 11]. We will show in Section 3 that the conditions of having weak drift 0 and of having weak drift 0 absolutely are useful in the description of $\mathfrak{R}(\Lambda_h^*)$ and $\mathfrak{R}^0(\Lambda_h^*)$. In Section 4 a similar fact will be shown in the description of $\mathfrak{R}_\infty(\Lambda_h)$ and $\mathfrak{R}_\infty(\Lambda_h^*)$, the limit of the ranges of iterations of Λ_h and Λ_h^* , respectively.

In Section 5, we will give a necessary and sufficient condition for a Lévy process $\{X_t^{(\mu)}\}$ on \mathbb{R}^d to satisfy the weak law of large numbers as $t \rightarrow \infty$ is that μ has weak mean and satisfies the condition $\lim_{t \rightarrow \infty} t \int_{|x| > t} \nu_\mu(dx) = 0$. On the other hand we will show, using the inversion, that a Lévy process $\{X_t^{(\mu)}\}$ without Gaussian part satisfies the weak version of Shtatland's theorem [12] (that is, $t^{-1}X_t^{(\mu)}$ converges in law to some constant as $t \downarrow 0$) if and only if the Lévy process $\{X_t^{(\mu')}\}$ satisfies the weak law of large numbers. Thus it will be shown that a necessary and sufficient condition for a Lévy process $\{X_t^{(\mu)}\}$ without Gaussian part to satisfy the weak version of Shtatland's theorem is that μ has a weak drift and satisfies $\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_{|x| \leq \varepsilon} |x|^2 \nu_\mu(dx) = 0$.

2. WEAK DRIFTS OF INFINITELY DIVISIBLE DISTRIBUTIONS

We say that $\mu \in ID$ has weak mean in \mathbb{R}^d if

$$\int_{1 < |x| \leq a} x \nu_\mu(dx) \text{ is convergent in } \mathbb{R}^d \text{ as } a \rightarrow \infty. \quad (2.1)$$

We say that $\mu \in ID$ has weak mean m_μ if (2.1) holds and $\hat{\mu}(z)$ satisfies

$$\hat{\mu}(z) = \exp \left[-\frac{1}{2} \langle z, A_\mu z \rangle + \lim_{a \rightarrow \infty} \int_{|x| \leq a} (e^{i \langle z, x \rangle} - 1 - i \langle z, x \rangle) \nu_\mu(dx) + i \langle m_\mu, z \rangle \right]. \quad (2.2)$$

If $\mu \in ID$ has mean m_μ , then μ has weak mean m_μ . If $\mu \in ID$ has weak mean m_μ , then $m_\mu = \gamma_\mu + \lim_{a \rightarrow \infty} \int_{1 < |x| \leq a} x \nu_\mu(dx)$. Let $(\bar{\nu}_\mu(dr), \lambda_r^\mu(d\xi))$ be a spherical decomposition of ν_μ , that is,

$$\nu_\mu(B) = \int_{\mathbb{R}_+^\circ} \bar{\nu}_\mu(dr) \int_S 1_B(r\xi) \lambda_r^\mu(d\xi), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where $\bar{\nu}_\mu$ is a σ -finite measure on $\mathbb{R}_+^\circ = (0, \infty)$ with $\bar{\nu}_\mu(\mathbb{R}_+^\circ) \geq 0$ and $\{\lambda_r^\mu : r \in \mathbb{R}_+^\circ\}$ is a measurable family of σ -finite measures on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $\lambda_r^\mu(S) > 0$ (S is the unit sphere if $d \geq 2$ or the two-point set $\{1, -1\}$ if $d = 1$); the decomposition is unique up to a change to $(c(r)\bar{\nu}_\mu(dr), c(r)^{-1}\lambda_r^\mu(d\xi))$ with a positive, finite, measurable

function $c(r)$ on \mathbb{R}_+° . We say that $\mu \in ID$ has weak mean in \mathbb{R}^d absolutely if

$$\int_{(1,\infty)} r \bar{\nu}_\mu(dr) \left| \int_S \xi \lambda_r^\mu(d\xi) \right| < \infty. \quad (2.3)$$

We say that $\mu \in ID$ has weak mean m_μ absolutely if (2.3) holds and μ has weak mean m_μ . These notions are introduced in [9].

Now we give the following definitions.

Definition 2.1. We say that $\mu \in ID$ has *weak drift* in \mathbb{R}^d if

$$\int_{\varepsilon < |x| \leq 1} x \nu_\mu(dx) \text{ is convergent in } \mathbb{R}^d \text{ as } \varepsilon \downarrow 0. \quad (2.4)$$

We say that $\mu \in ID$ has weak drift γ_μ^0 if (2.4) holds and

$$\hat{\mu}(z) = \exp \left[-\frac{1}{2} \langle z, A_\mu z \rangle + \lim_{\varepsilon \downarrow 0} \int_{|x| > \varepsilon} (e^{i \langle z, x \rangle} - 1) \nu_\mu(dx) + i \langle \gamma_\mu^0, z \rangle \right].$$

Property (2.4) is equivalent to saying that, for each $z \in \mathbb{R}^d$, $\int_{|x| > \varepsilon} (e^{i \langle z, x \rangle} - 1) \nu_\mu(dx)$ is convergent in \mathbb{C} as $\varepsilon \downarrow 0$.

Definition 2.2. We say that $\mu \in ID$ has *weak drift* in \mathbb{R}^d *absolutely* if

$$\int_{(0,1]} r \bar{\nu}_\mu(dr) \left| \int_S \xi \lambda_r^\mu(d\xi) \right| < \infty, \quad (2.5)$$

where $(\bar{\nu}_\mu(dr), \lambda_r^\mu(d\xi))$ is a spherical decomposition of ν_μ . We say that $\mu \in ID$ has weak drift γ_μ^0 absolutely if (2.5) holds and μ has weak drift γ_μ^0 .

We remark that, if (2.5) holds for some spherical decomposition of ν_μ , then it holds for any spherical decomposition of ν_μ .

Proposition 2.3. If $\mu \in ID$ has weak drift γ_μ^0 , then $\gamma_\mu^0 = \gamma_\mu - \lim_{\varepsilon \downarrow 0} \int_{\varepsilon < |x| \leq 1} x \nu_\mu(dx)$.

Proof. Compare (1.1) and (2.2). □

The following result is basic in this paper.

Theorem 2.4. Let $\mu \in ID_0$. Then the inversion μ' of μ has weak drift in \mathbb{R}^d if and only if μ has weak mean in \mathbb{R}^d . The inversion μ' has weak drift in \mathbb{R}^d absolutely if and only if μ has weak mean in \mathbb{R}^d absolutely. If μ' has weak drift $\gamma_{\mu'}^0$, then $\gamma_{\mu'}^0 = -m_\mu$, where m_μ is the weak mean of μ .

Since $\mu'' = \mu$, we can interchange “weak drift” and “weak mean” in the second and third sentences of the theorem.

Proof of Theorem 2.4. We have

$$\int_{\mathbb{R}^d} h(x) \nu_{\mu'}(dx) = \int_{\mathbb{R}^d} h(|x|^{-2}x) |x|^2 \nu_\mu(dx)$$

for any \mathbb{R}^d -valued function $h(x)$ on \mathbb{R}^d satisfying $\int |h(x)|\nu_{\mu'}(dx) = \int |h(|x|^{-2}x)| |x|^2 \nu_{\mu}(dx) < \infty$, as in the proof of Proposition 2.1 of [11]. Hence

$$\int_{\varepsilon < |x| \leq 1} x\nu_{\mu'}(dx) = \int_{1 \leq |x| < 1/\varepsilon} x\nu_{\mu}(dx). \quad (2.6)$$

Thus the second sentence of the theorem follows. The fourth sentence also follows, since

$$\begin{aligned} \gamma_{\mu'}^0 &= \gamma_{\mu'} - \lim_{\varepsilon \downarrow 0} \int_{\varepsilon < |x| \leq 1} x\nu_{\mu'}(dx) = -\gamma_{\mu} + \int_{|x|=1} x\nu_{\mu}(dx) - \lim_{a \rightarrow \infty} \int_{1 \leq |x| < a} x\nu_{\mu}(dx) \\ &= -\gamma_{\mu} - \lim_{a \rightarrow \infty} \int_{1 < |x| \leq a} x\nu_{\mu}(dx) = -m_{\mu}. \end{aligned}$$

Let $(\bar{\nu}_{\mu}(dr), \lambda_r^{\mu}(d\xi))$ be a spherical decomposition of ν_{μ} . Define

$$\bar{\nu}^{\sharp}(E) = \int_{\mathbb{R}_+^{\circ}} 1_E(r^{-1})r^2\bar{\nu}_{\mu}(dr), \quad E \in \mathcal{B}(\mathbb{R}_+^{\circ}).$$

Then

$$\nu_{\mu'}(B) = \int_{\mathbb{R}_+^{\circ}} \bar{\nu}_{\mu}(dr) \int_S 1_B(r^{-1}\xi)r^2\lambda_r^{\mu}(d\xi) = \int_{\mathbb{R}_+^{\circ}} \bar{\nu}^{\sharp}(dr) \int_S 1_B(r\xi)\lambda_{r^{-1}}^{\mu}(d\xi).$$

Hence $\nu_{\mu'}$ has a spherical decomposition $(\bar{\nu}_{\mu'}(dr), \lambda_r^{\mu'}(d\xi))$ with $\bar{\nu}_{\mu'} = \bar{\nu}^{\sharp}$ and $\lambda_r^{\mu'} = \lambda_{r^{-1}}^{\mu}$. It follows that

$$\begin{aligned} \int_{(0,1]} r\bar{\nu}_{\mu'}(dr) \left| \int_S \xi \lambda_r^{\mu'}(d\xi) \right| &= \int_{(0,1]} r\bar{\nu}^{\sharp}(dr) \left| \int_S \xi \lambda_{r^{-1}}^{\mu}(d\xi) \right| \\ &= \int_{[1,\infty)} r^{-1}r^2\bar{\nu}_{\mu}(dr) \left| \int_S \xi \lambda_r^{\mu}(d\xi) \right|, \end{aligned}$$

which yields the third sentence of the theorem. \square

Proposition 2.5. *Let $\mu \in ID$. If μ is symmetric, then μ has weak drift 0 absolutely.*

Proof. Assume that μ is symmetric, that is, $\mu(-B) = \mu(B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. Then $\widehat{\mu}(z)$ is real. Hence ν_{μ} is symmetric and $\gamma_{\mu} = 0$. Thus $\int_{\varepsilon < |x| \leq 1} x\nu_{\mu}(dx) = 0$. Hence μ has weak drift and it follows from Proposition 2.3 that the weak drift is 0. Let $(\bar{\nu}_{\mu}(dr), \lambda_r^{\mu}(d\xi))$ be a spherical decomposition of ν_{μ} . The symmetry of ν_{μ} yields that, for $\bar{\nu}_{\mu}$ -a. e. r , λ_r^{μ} is symmetric, so that $\int_S \xi \lambda_r^{\mu}(d\xi) = 0$. Hence (2.5) holds. \square

As a digression we mention a property of weak drift, which is applicable to characterization of strict 1-stability. We say that the Lévy measure ν_{μ} of $\mu \in ID$ is of polar product type if there are a finite measure λ_{μ} on S and a σ -finite measure $\bar{\nu}_{\mu}$ on \mathbb{R}_+° such that $\nu_{\mu}(B) = \int_S \lambda_{\mu}(d\xi) \int_{\mathbb{R}_+^{\circ}} 1_B(r\xi)\bar{\nu}_{\mu}(dr)$ for $B \in \mathcal{B}(\mathbb{R}^d)$.

Proposition 2.6. *Let $\mu \in ID$ with ν_{μ} of polar product type. Assume that $\int_{|x| \leq 1} |x|\nu_{\mu}(dx) = \infty$. Then the following five conditions are equivalent.*

- (i) μ has weak drift in \mathbb{R}^d .
- (ii) $\lim_{\varepsilon \downarrow 0} \int_{\varepsilon < |x| \leq 1} x \nu_\mu(dx) = 0$.
- (iii) μ has weak drift in \mathbb{R}^d absolutely.
- (iv) μ has weak drift absolutely and $\lim_{\varepsilon \downarrow 0} \int_{\varepsilon < |x| \leq 1} x \nu_\mu(dx) = 0$.
- (v) λ_μ in the definition of polar product type satisfies $\int_S \xi \lambda_\mu(d\xi) = 0$.

Proof. The implications (iv) \Rightarrow (ii) \Rightarrow (i) and (iv) \Rightarrow (iii) \Rightarrow (i) are obvious. We have $\int_{\varepsilon < |x| \leq 1} x \nu_\mu(dx) = \int_S \xi \lambda_\mu(d\xi) \int_{(\varepsilon, 1]} r \bar{\nu}_\mu(dr)$, since λ_μ is of polar product type. Moreover, since $\int_{|x| \leq 1} |x| \nu_\mu(dx) = \infty$, we have $\int_{(0, 1]} r \bar{\nu}_\mu(dr) = \infty$. It follows that (i) implies (v). As $(\bar{\nu}_\mu(dr), \lambda_\mu(d\xi))$ gives a spherical decomposition of ν_μ , (v) implies (iv). This proof is similar to that of Proposition 3.15 of [9]. \square

Example 2.7. For $0 < \alpha < 2$ the Lévy measure ν_μ of an α -stable distribution μ on \mathbb{R}^d is of polar product type in the form $\nu_\mu(B) = \int_S \lambda_\mu(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r\xi) r^{-1-\alpha} dr$. The condition $\int_{|x| \leq 1} |x| \nu_\mu(dx) = \infty$ is satisfied if and only if μ is nontrivial (that is, not a δ -measure) and $1 \leq \alpha < 2$. Let μ be a 1-stable distribution on \mathbb{R}^d . If $d = 1$, then μ is strictly 1-stable if and only if ν_μ is symmetric. If $d \geq 2$, then symmetry of ν_μ implies strict 1-stability of μ , but the Lévy measure of strictly 1-stable distribution is not always symmetric. A necessary and sufficient condition for μ to be strictly 1-stable is that $\int_S \xi \lambda_\mu(d\xi) = 0$ (see [7]). Hence Proposition 2.6 gives equivalent characterizations of strict 1-stability for a nontrivial 1-stable distribution. Similar characterizations using weak mean are given in Example 3.16 of [9].

3. RANGES OF CONJUGATES OF SOME STOCHASTIC INTEGRAL MAPPINGS

Conjugates of stochastic integral mappings are introduced in [11] in the following way. A function $h(u)$ is said to satisfy Condition (C) if there are a_h and b_h with $0 \leq a_h < b_h \leq \infty$ such that h is defined on (a_h, b_h) , positive, and measurable, and

$$\min \left\{ \int_{a_h}^{b_h} h(u) u^2 du, \int_{a_h}^{b_h} h(u) du \right\} < \infty.$$

For any h satisfying Condition (C) we define a function h^* as $a_{h^*} = 1/b_h$, $b_{h^*} = 1/a_h$, and

$$h^*(u) = h(u^{-1})u^{-4}, \quad u \in (a_{h^*}, b_{h^*}).$$

Then h^* automatically satisfies Condition (C) and we have $(h^*)^* = h$.

Let h be a function satisfying Condition (C). Define a strictly decreasing continuous function $g_h(t)$ as $g_h(t) = \int_t^{b_h} h(u) du$ for $t \in (a_h, b_h)$ and let $c_h = g_h(a_h+)$. Let $t = f_h(s)$, $0 < s < c_h$, be the inverse function of $s = g_h(t)$, $a_h < t < b_h$. Then $f_h(s)$ is a strictly decreasing continuous function with $f_h(0+) = b_h$ and $f_h(c_h-) = a_h$. For all $\rho \in ID$, the stochastic integral $\int_p^q f_h(s) dX_s^{(\rho)}$ with respect to a Lévy process $\{X_s^{(\rho)}\}$ with distribution ρ at time 1 is defined either for $0 \leq p < q < c_h = \infty$ or for

$0 < p < q \leq c_h < \infty$. If h satisfies $\int_{a_h}^{b_h} h(u)u^2 du < \infty$, then the stochastic integral mapping Φ_{f_h} is defined as $\Phi_{f_h}\rho = \mathcal{L}\left(\int_0^{c_h^-} f_h(s)dX_s^{(\rho)}\right)$ whenever $\int_0^{c_h^-} f_h(s)dX_s^{(\rho)}$ is definable. If h satisfies $\int_{a_h}^{b_h} h(u)du < \infty$, then $c_h < \infty$ and Φ_{f_h} is defined as $\Phi_{f_h}\rho = \mathcal{L}\left(\int_{0+}^{c_h} f_h(s)dX_s^{(\rho)}\right)$ whenever $\int_{0+}^{c_h} f_h(s)dX_s^{(\rho)}$ is definable. The mapping Φ_{f_h} is written as Λ_h . Given a function h satisfying Condition (C), we call Λ_{h^*} the conjugate of Λ_h and write $\Lambda_h^* = \Lambda_{h^*} = \Phi_{f_{h^*}}$. Since $(h^*)^* = h$, the conjugate of Λ_h^* equals Λ_h . In the analysis of $\mathfrak{D}(\Lambda_h)$ and $\mathfrak{R}(\Lambda_h)$ we use the following restriction and extension of Λ_h . We say that $\Lambda_h\rho$ is absolutely definable if $\int_0^{c_h} |\log \widehat{\rho}(f_h(s)z)|ds < \infty$ for $z \in \mathbb{R}^d$. Let $\mathfrak{D}^0(\Lambda_h) = \{\rho \in \mathfrak{D}(\Lambda_h) : \Lambda_h\rho \text{ is absolutely definable}\}$ and $\mathfrak{R}^0(\Lambda_h) = \{\Lambda_h\rho : \rho \in \mathfrak{D}^0(\Lambda_h)\}$. If h satisfies $\int_{a_h}^{b_h} h(u)u^2 du < \infty$, then we say that $\Lambda_h\rho$ is essentially definable if, for some \mathbb{R}^d -valued function $k(q)$ for $0 < q < c_h$ and some \mathbb{R}^d -valued random variable Y , $\int_0^q f_h(s)dX_s^{(\rho)} - k(q)$ converges to Y in probability as $q \uparrow c_h$. If h satisfies $\int_{a_h}^{b_h} h(u)du < \infty$, then $c_h < \infty$ and we say that $\Lambda_h\rho$ is essentially definable if, for some \mathbb{R}^d -valued function $k(p)$ for $0 < p < c_h$ and some \mathbb{R}^d -valued random variable Y , $\int_p^{c_h} f_h(s)dX_s^{(\rho)} - k(p)$ converges to Y in probability as $p \downarrow 0$. Let $\mathfrak{D}^e(\Lambda_h) = \{\rho \in ID : \Lambda_h\rho \text{ is essentially definable}\}$ and let $\mathfrak{R}^e(\Lambda_h)$ be the class of $\mu = \mathcal{L}(Y)$ where all $\rho \in \mathfrak{D}^e(\Lambda_h)$ and all k and Y that can be chosen in the definition of essential definability of $\Lambda_h\rho$ are taken into account. Notice that $\mathfrak{D}^0(\Lambda_h) \subset \mathfrak{D}(\Lambda_h) \subset \mathfrak{D}^e(\Lambda_h)$ and $\mathfrak{R}^0(\Lambda_h) \subset \mathfrak{R}(\Lambda_h) \subset \mathfrak{R}^e(\Lambda_h)$.

In this section we are interested in the mappings $\bar{\Phi}_{p,\alpha}$ and $\Psi_{\alpha,\beta}$ as a continuation of [11]. We will also mention the mapping $\Lambda_{q,\alpha}$ in Sections 4 and 5. Their definitions are as follows.

1. Given $p > 0$ [resp. $q > 0$] and $-\infty < \alpha < 2$, let $a_h = 0$, $b_h = 1$, and $h(u) = \Gamma(p)^{-1}(1-u)^{p-1}u^{-\alpha-1}$ [resp. $h(u) = \Gamma(q)^{-1}(-\log u)^{q-1}u^{-\alpha-1}$]. Then h satisfies Condition (C) with $\int_0^1 h(u)u^2 du < \infty$ and c_h is finite and infinite according as $\alpha < 0$ or $\alpha \geq 0$; h^* satisfies condition (C) with $\int_1^\infty h^*(u)du < \infty$ and hence $c_{h^*} < \infty$. The mapping Λ_h is denoted by $\bar{\Phi}_{p,\alpha}$ [resp. $\Lambda_{q,\alpha}$].

2. Given $-\infty < \alpha < 2$ and $\beta > 0$, let $a_h = 0$, $b_h = \infty$, and $h(u) = u^{-\alpha-1}e^{-u^\beta}$. Then h satisfies Condition (C) with $\int_0^\infty h(u)u^2 du < \infty$ and c_h is finite and infinite according as $\alpha < 0$ or $\alpha \geq 0$; h^* satisfies condition (C) with $\int_0^\infty h^*(u)du < \infty$ and hence $c_{h^*} < \infty$. The mapping Λ_h is denoted by $\Psi_{\alpha,\beta}$.

Let Λ_h equal $\bar{\Phi}_{p,\alpha}$ or $\Psi_{\alpha,\beta}$. The domains \mathfrak{D} , \mathfrak{D}^0 , \mathfrak{D}^e and the ranges \mathfrak{R} , \mathfrak{R}^0 , \mathfrak{R}^e of both Λ_h and Λ_h^* are given description in [11] if $\alpha \neq 1$. In the case $\alpha = 1$ the domains of Λ_h and Λ_h^* and the ranges $\mathfrak{R}(\Lambda_h)$, $\mathfrak{R}^0(\Lambda_h)$, $\mathfrak{R}^e(\Lambda_h)$, and $\mathfrak{R}^e(\Lambda_h^*)$ are described in [11], but $\mathfrak{R}(\Lambda_h^*)$ and $\mathfrak{R}^0(\Lambda_h^*)$ are not treated. Now we handle them, using the notion of weak drift. In the description of $\mathfrak{R}^e(\Lambda_h)$ and $\mathfrak{R}^e(\Lambda_h^*)$, we need to use some notions. A Lévy measure ν_μ is said to have a radial decomposition (rad. dec.)

$(\lambda(d\xi), \nu_\xi(dr))$ if $\nu_\mu(B) = \int_S \lambda(d\xi) \int_{\mathbb{R}_+^\circ} 1_B(r\xi) \nu_\xi(dr)$, $B \in \mathcal{B}(\mathbb{R}^d)$, where λ is a σ -finite measure on S with $\lambda(S) \geq 0$ and $\{\nu_\xi(dr) : \xi \in S\}$ is a measurable family of σ -finite measures on \mathbb{R}_+° with $\nu_\xi(\mathbb{R}_+^\circ) > 0$; the decomposition is unique up to a change to $(c(\xi)\lambda(d\xi), c(\xi)^{-1}\nu_\xi(dr))$ with a positive, finite, measurable function $c(\xi)$ on S . A $[0, \infty]$ -valued function $\varphi(u)$ on \mathbb{R}_+° is said to be monotone of order $p > 0$ if $\varphi(u)$ is locally integrable on \mathbb{R}_+° and there is a locally finite measure σ on \mathbb{R}_+° such that $\varphi(u) = \Gamma(p)^{-1} \int_{(u, \infty)} (r - u)^{p-1} \sigma(dr)$ for $u \in \mathbb{R}_+^\circ$. A function $\varphi(u)$ on \mathbb{R}_+° is said to be completely monotone if it is monotone of order p for every $p > 0$. In Theorem 4.4 and (4.23) of [11] it is shown that if $\Lambda_h = \bar{\Phi}_{p,1}$, then

$$\begin{aligned} \mathfrak{R}^e(\Lambda_h) &= \{\mu \in ID : \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{-2}k_\xi(u)du) \text{ such that } k_\xi(u) \\ &\quad \text{is measurable in } (\xi, u) \text{ and monotone of order } p \text{ in } u \in \mathbb{R}_+^\circ\}, \end{aligned} \quad (3.1)$$

$$\begin{aligned} \mathfrak{R}^e(\Lambda_h^*) &= \{\mu \in ID_0 : \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{-2}k_\xi(u^{-1})du) \text{ such that } k_\xi(v) \\ &\quad \text{is measurable in } (\xi, v) \text{ and monotone of order } p \text{ in } v \in \mathbb{R}_+^\circ\}. \end{aligned} \quad (3.2)$$

In Theorem 4.6 of [11] it is shown that if $\Lambda_h = \Psi_{1,\beta}$, then

$$\begin{aligned} \mathfrak{R}^e(\Lambda_h) &= \{\mu \in ID : \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{-2}k_\xi(u^\beta)du) \text{ such that } k_\xi(v) \\ &\quad \text{is measurable in } (\xi, v) \text{ and completely monotone in } v \in \mathbb{R}_+^\circ\}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} \mathfrak{R}^e(\Lambda_h^*) &= \{\mu \in ID_0 : \nu_\mu \text{ has a rad. dec. } (\lambda(d\xi), u^{-2}k_\xi(u^{-\beta})du) \text{ such that } k_\xi(v) \\ &\quad \text{is measurable in } (\xi, v) \text{ and completely monotone in } v \in \mathbb{R}_+^\circ\}. \end{aligned} \quad (3.4)$$

Our result is as follows.

Theorem 3.1. *Let $\Lambda_h = \bar{\Phi}_{p,1}$ with $p > 0$ or $\Lambda_h = \Psi_{1,\beta}$ with $\beta > 0$. Then,*

$$\mathfrak{R}(\Lambda_h) = \{\mu \in \mathfrak{R}^e(\Lambda_h) : \mu \text{ has weak mean } 0\}, \quad (3.5)$$

$$\mathfrak{R}^0(\Lambda_h) = \{\mu \in \mathfrak{R}^e(\Lambda_h) : \mu \text{ has weak mean } 0 \text{ absolutely}\}, \quad (3.6)$$

$$\mathfrak{R}(\Lambda_h^*) = \{\mu \in \mathfrak{R}^e(\Lambda_h^*) : \mu \text{ has weak drift } 0\}, \quad (3.7)$$

$$\mathfrak{R}^0(\Lambda_h^*) = \{\mu \in \mathfrak{R}^e(\Lambda_h^*) : \mu \text{ has weak drift } 0 \text{ absolutely}\}. \quad (3.8)$$

Proof. The assertions (3.5) and (3.6) are shown in Theorems 4.4 and 4.6 of [11]. In order to obtain (3.7) and (3.8) from these, we use the basic relations of conjugates of stochastic integral mappings with inversions given by

$$\mathfrak{R}(\Lambda_h^*)_0 = (\mathfrak{R}(\Lambda_h)_0)' \quad \mathfrak{R}^e(\Lambda_h^*)_0 = (\mathfrak{R}^e(\Lambda_h)_0)', \quad \mathfrak{R}^0(\Lambda_h^*)_0 = (\mathfrak{R}^0(\Lambda_h)_0)'$$

in Theorem 3.6 of [11]. We have

$$\begin{aligned} \mathfrak{R}(\Lambda_h^*)_0 &= \{\mu \in ID_0 : \mu' \in \mathfrak{R}(\Lambda_h)_0\} \\ &= \{\mu \in ID_0 : \mu' \in \mathfrak{R}^e(\Lambda_h)_0 \text{ and } \mu' \text{ has weak mean } 0\} \\ &= \{\mu \in ID_0 : \mu \in \mathfrak{R}^e(\Lambda_h^*)_0 \text{ and } \mu \text{ has weak drift } 0\} \end{aligned}$$

from (3.5), since μ' has weak mean 0 if and only if μ has weak drift 0 by virtue of Theorem 2.4. This proves (3.7), since $\mathfrak{R}^e(\Lambda_h^*)_0 = \mathfrak{R}^e(\Lambda_h^*)$ from (3.2) and (3.4). The proof of (3.8) is similarly obtained from (3.2), (3.4), and (3.6), using the fact in Theorem 2.4 that μ' has weak mean 0 absolutely if and only if μ has weak drift 0 absolutely. \square

4. LIMITS OF SOME NESTED CLASSES

For a stochastic integral mapping Φ_f its iterations Φ_f^n , $n = 1, 2, \dots$, are defined as $\Phi_f^1 = \Phi_f$ and $\Phi_f^{n+1}\rho = \Phi_f(\Phi_f^n\rho)$ with $\mathfrak{D}(\Phi_f^n) = \{\rho \in \mathfrak{D}(\Phi_f^n) : \Phi_f^n\rho \in \mathfrak{D}(\Phi_f)\}$. Then we get nested classes $ID \supset \mathfrak{R}(\Phi_f) \supset \mathfrak{R}(\Phi_f^2) \supset \dots$. Let $\mathfrak{R}_\infty(\Phi_f) = \bigcap_{n=1}^\infty \mathfrak{R}(\Phi_f^n)$, the limit of the nested classes. The class $\mathfrak{R}_\infty(\Phi_f)$ is possibly identical for different functions f . For example, it was shown in [4] that $\mathfrak{R}_\infty(\Phi_f)$ equals the class L_∞ for many stochastic integral mappings Φ_f known at that time. Here L_∞ is the class of completely selfdecomposable distributions on \mathbb{R}^d , which is the smallest class closed under convolution and weak convergence and containing all stable distributions on \mathbb{R}^d . A distribution $\mu \in ID$ belongs to L_∞ if and only if

$$\nu_\mu(B) = \int_{(0,2)} \Gamma_\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(r\xi) r^{-\beta-1} dr, \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where Γ_μ is a measure on $(0, 2)$ satisfying $\int_{(0,2)} (\beta^{-1} + (2 - \beta)^{-1}) \Gamma_\mu(d\beta) < \infty$ and $\{\lambda_\beta^\mu : \beta \in (0, 2)\}$ is a measurable family of probability measures on S . This representation of ν_μ is unique. For a Borel subset E of $(0, 2)$, L_∞^E denotes the class of $\mu \in L_\infty$ such that Γ_μ is concentrated on E .

We are interested in what classes appear as $\mathfrak{R}_\infty(\Lambda_h)$ and $\mathfrak{R}_\infty(\Lambda_h^*)$ for stochastic integral mappings Λ_h associated with functions h satisfying Condition (C). In [10, 11] the description of $\mathfrak{R}_\infty(\Lambda_h)$ and $\mathfrak{R}_\infty(\Lambda_h^*)$ is given for Λ_h equal to $\bar{\Phi}_{p,\alpha}$, $\Lambda_{q,\alpha}$, and $\Psi_{\alpha,1}$ with $\alpha \in (-\infty, 1) \cup (1, 2)$, $p \geq 1$, and $q > 0$. The description of $\mathfrak{R}_\infty(\Lambda_h)$ is also given in the case $\alpha = 1$, $p \geq 1$, and $q = 1$ in [10]. Actually $\Lambda_{1,\alpha} = \bar{\Phi}_{1,\alpha}$. Now let us treat $\mathfrak{R}_\infty(\Lambda_h^*)$ for Λ_h equal to $\bar{\Phi}_{p,1}$ and $\Psi_{1,1}$ with $p \geq 1$. Again the notion of weak drift is crucial.

Theorem 4.1. *Let $\Lambda_h = \bar{\Phi}_{p,1}$ with $p \geq 1$ or $\Lambda_h = \Psi_{1,1}$. Then*

$$\mathfrak{R}_\infty(\Lambda_h) = L_\infty^{(1,2)} \cap \{\mu \in ID : \mu \text{ has weak mean } 0\}, \quad (4.1)$$

$$\mathfrak{R}_\infty(\Lambda_h^*) = (L_\infty^{(0,1)})_0 \cap \{\mu \in ID : \mu \text{ has weak drift } 0\}. \quad (4.2)$$

Proof. The description (4.1) of $\mathfrak{R}_\infty(\Lambda_h)$ is shown in Theorem 1.1 of [10]. We have $\mathfrak{R}_\infty(\Lambda_h^*)_0 = (\mathfrak{R}_\infty(\Lambda_h)_0)'$ in Theorem 6.3 of [11], and $((L_\infty^{(1,2)})_0)' = (L_\infty^{(0,1)})_0$ obtained from Proposition 6.1 of [11]. Hence $\mathfrak{R}_\infty(\Lambda_h^*)_0$ is identical with the right-hand side of (4.1) by virtue of Theorem 2.4. It remains to see $\mathfrak{R}_\infty(\Lambda_h^*) = \mathfrak{R}_\infty(\Lambda_h^*)_0$. But, since $f_{h^*}(s) \asymp s^{-1}$ as $s \downarrow 0$ by (4.3) and (4.38) of [11], we have $\int_0^{c_{h^*}} f_{h^*}(s)^2 ds = \infty$, and

hence $\mathfrak{D}(\Lambda_h^*) \subset ID_0$ and $\mathfrak{R}(\Lambda_h^*) \subset ID_0$ as in Proposition 3.8 of [11]. Now it follows that $\mathfrak{R}_\infty(\Lambda_h^*) \subset ID_0$. \square

5. WEAK LAW OF LARGE NUMBERS AND WEAK VERSION OF SHTATLAND'S THEOREM

Shtatland [12] proves that, for a Lévy process $\{X_t^{(\mu)} : t \geq 0\}$ on \mathbb{R} , $\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} X_\varepsilon^{(\mu)} = c$ almost surely for $c \in \mathbb{R}$ if and only if μ has drift c and no Gaussian part. The “if” part is easily extended to \mathbb{R}^d ; see Theorem 43.20 of [7]. A weaker conclusion is that, for a Lévy process $\{X_t^{(\mu)} : t \geq 0\}$ on \mathbb{R}^d , if $\mu \in ID_0$ and μ has drift c , then $\mathcal{L}(\varepsilon^{-1} X_\varepsilon^{(\mu)}) \rightarrow \delta_c$ as $\varepsilon \downarrow 0$. The following fact shows its connection with the weak law of large numbers through inversion.

Theorem 5.1. *Let $\mu \in ID_0$ and $c \in \mathbb{R}^d$. Then $\mathcal{L}(\varepsilon^{-1} X_\varepsilon^{(\mu)}) \rightarrow \delta_c$ as $\varepsilon \downarrow 0$ if and only if $\mathcal{L}(t^{-1} X_t^{(\mu)}) \rightarrow \delta_{-c}$ as $t \rightarrow \infty$.*

Proof. The core of the proof is the formula $(T_b \mu)' = (T_{b^{-1}}(\mu'))^{b^2}$ for $b > 0$ and $\mu \in ID_0$ proved in Proposition 2.4 of [11], where T_b is the dilation $(T_b \mu)(B) = \int_{\mathbb{R}^d} 1_B(bx) \mu(dx)$ and $\mu^t = \mathcal{L}(X_t^{(\mu)})$. Notice that $T_b(\mu^t) = (T_b \mu)^t$. We also use properties of the inversion in Proposition 2.1 (vi), (viii), and (ix) of [11]. Assume that $\mathcal{L}(\varepsilon^{-1} X_\varepsilon^{(\mu)}) \rightarrow \delta_c$ as $\varepsilon \downarrow 0$. Then $(\mathcal{L}(\varepsilon^{-1} X_\varepsilon^{(\mu)}))' \rightarrow \delta'_c = \delta_{-c}$. We have

$$(\mathcal{L}(\varepsilon^{-1} X_\varepsilon^{(\mu)}))' = (T_{\varepsilon^{-1}}(\mu^\varepsilon))' = (T_\varepsilon((\mu^\varepsilon)'))^{\varepsilon^{-2}} = (T_\varepsilon((\mu')^\varepsilon))^{\varepsilon^{-2}} = (T_\varepsilon(\mu'))^{\varepsilon^{-1}},$$

which is equal to $\mathcal{L}(t^{-1} X_t^{(\mu')})$ for $t = \varepsilon^{-1}$. The converse is similar. \square

Necessary and sufficient conditions for the weak law of large numbers for Lévy processes are as follows.

Theorem 5.2. *Let $\mu \in ID$ and $c \in \mathbb{R}^d$. The following three statements are equivalent.*

- (i) *The Lévy process $\{X_t^{(\mu)} : t \geq 0\}$ on \mathbb{R}^d satisfies $\mathcal{L}(t^{-1} X_t^{(\mu)}) \rightarrow \delta_c$ as $t \rightarrow \infty$.*
- (ii) *The distribution μ has weak mean c and*

$$\lim_{t \rightarrow \infty} t \int_{|x| > t} \nu_\mu(dx) = 0. \tag{5.1}$$

- (iii) *The distribution μ satisfies*

$$\lim_{t \rightarrow \infty} \int_{|x| \leq t} x \mu(dx) = c, \tag{5.2}$$

$$\lim_{t \rightarrow \infty} t \int_{|x| > t} \mu(dx) = 0. \tag{5.3}$$

Proof. The equivalence of (i) and (ii) is as follows. It is convenient to use the Lévy–Khintchine representation in the form

$$\widehat{\mu}(z) = \exp \left[-\frac{1}{2} \langle z, A_\mu z \rangle + \int_{\mathbb{R}^d} (e^{i \langle z, x \rangle} - 1 - i \langle z, x \rangle c(x)) \nu_\mu(dx) + i \langle \gamma_\mu^\sharp, z \rangle \right]$$

for $\mu \in ID$ with $c(x) = 1_{\{|x| \leq 1\}}(x) + |x|^{-1} 1_{\{|x| > 1\}}(x)$ adopted by Rajput and Rosinski [6] and Kwapień and Woyczyński [2], as in (2.4) of [11]. Let us call $\gamma_\mu^\#$ the $\#$ -location parameter of μ ; $\gamma_\mu^\#$ is related to the location parameter γ_μ in (1.1) as

$$\gamma_\mu^\# = \gamma_\mu + \int_{|x| > 1} |x|^{-1} x \nu_\mu(dx). \quad (5.4)$$

The dilation $T_b\mu$ of μ with $0 < b < 1$ has triplet

$$A_{T_b\mu} = b^2 A_\mu, \quad \nu_{T_b\mu} = T_b \nu_\mu, \quad \gamma_{T_b\mu} = b\gamma_\mu + b \int_{1 < |x| \leq b^{-1}} x \nu_\mu(dx)$$

as in (2.7) of [11]. Hence, for $t > 1$, $\mathcal{L}(t^{-1}X_t^{(\mu)}) = T_{t^{-1}}(\mu^t) = (T_{t^{-1}}\mu)^t$ has Gaussian covariance matrix $t^{-1}A_\mu$, Lévy measure $tT_{t^{-1}}\nu_\mu$, and $\#$ -location parameter

$$\gamma_\mu + \int_{1 < |x| \leq t} x \nu_\mu(dx) + t \int_{|x| > t} |x|^{-1} x \nu_\mu(dx)$$

from (5.4). Since $c(x)$ is continuous on \mathbb{R}^d , we can use Theorem 8.7 of [7] and see that $\mathcal{L}(t^{-1}X_t^{(\mu)}) \rightarrow \delta_c$ as $t \rightarrow \infty$ if and only if

$$t\nu_\mu(tB) \rightarrow 0 \text{ for all } B \in \mathcal{B}(\mathbb{R}^d) \text{ such that } 0 \text{ is not in the closure of } B, \quad (5.5)$$

$$\lim_{\eta \downarrow 0} \limsup_{t \rightarrow \infty} \left(\langle z, t^{-1}A_\mu z \rangle + t \int_{|x| < \eta} \langle z, t^{-1}x \rangle^2 \nu_\mu(dx) \right) = 0 \text{ for } z \in \mathbb{R}^d, \quad (5.6)$$

and

$$\gamma_\mu + \int_{1 < |x| \leq t} x \nu_\mu(dx) + t \int_{|x| > t} |x|^{-1} x \nu_\mu(dx) \rightarrow c. \quad (5.7)$$

Condition (5.5) is the same as $t \int_{|x| > t\eta} \nu_\mu(dx) \rightarrow 0$ for $\eta > 0$, which is equivalent to (5.1). Condition (5.6) is always satisfied. If (5.1) holds, then $t \int_{|x| > t} |x|^{-1} x \nu_\mu(dx) \rightarrow 0$ and condition (5.7) is expressed as μ has weak mean c .

Next, let us prove the equivalence of (i) and (iii). If (i) holds, then $n^{-1}X_n^{(\mu)} \rightarrow c$ in probability as $n = 1, 2, \dots \rightarrow \infty$. Since $\{X_n^{(\mu)}\}$ is the sum of i.i.d. random variables, we see from the theorem in p. 565 of Feller [1] and Theorem 36.4 of [7] that (i) implies (iii). These theorems also show that (iii) implies that $n^{-1}X_n^{(\mu)} \rightarrow c$ in probability as $n \rightarrow \infty$. Statement (i) follows from this, since, for $n \leq t < n+1$,

$$\begin{aligned} t^{-1}X_t^{(\mu)} &= t^{-1}(X_t^{(\mu)} - X_n^{(\mu)}) + (t^{-1} - n^{-1})X_n^{(\mu)} + n^{-1}X_n^{(\mu)}, \\ t^{-1}|X_t^{(\mu)} - X_n^{(\mu)}| &\stackrel{\text{law}}{=} t^{-1}|X_{t-n}^{(\mu)}| \leq t^{-1} \sup_{s \leq 1} |X_s^{(\mu)}| \rightarrow 0 \quad \text{a.s., } t \rightarrow \infty, \end{aligned}$$

and

$$|(t^{-1} - n^{-1})X_n^{(\mu)}| \leq n^{-2}|X_n^{(\mu)}| \rightarrow 0 \quad \text{in probability, } t \rightarrow \infty.$$

This finishes the proof. \square

As to the inversion version of Theorem 5.2, we can prove the equivalence of the analogues of (i) and (ii).

Theorem 5.3. *Let $\mu \in ID_0$ and $c \in \mathbb{R}^d$. The following two statements are equivalent.*

- (i) *The Lévy process $\{X_t^{(\mu)} : t \geq 0\}$ on \mathbb{R}^d satisfies $\mathcal{L}(\varepsilon^{-1}X_\varepsilon^{(\mu)}) \rightarrow \delta_c$ as $\varepsilon \downarrow 0$.*
- (ii) *The distribution μ has weak drift c and*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_{|x| \leq \varepsilon} |x|^2 \nu_\mu(dx) = 0. \quad (5.8)$$

Proof. Combine Theorem 5.1 with the equivalence of (i) and (ii) of Theorem 5.2. Then, $\mathcal{L}(\varepsilon^{-1}X_\varepsilon^{(\mu)}) \rightarrow \delta_c$ as $\varepsilon \downarrow 0$ if and only if μ' has weak mean $-c$ and $t \int_{|x| > t} \nu_{\mu'}(dx) \rightarrow 0$ as $t \rightarrow \infty$. Use Theorem 2.4 and that $t \int_{|x| > t} \nu_{\mu'}(dx) = t \int_{|x| < t^{-1}} |x|^2 \nu_\mu(dx)$. Now we see that our assertion is true. \square

We give two final remarks concerning the conditions (5.1) and (5.8).

1. If $\mu \in ID$ has mean, then μ satisfies (5.1), since $t \int_{|x| > t} \nu_\mu(dx) \leq \int_{|x| > t} |x| \nu_\mu(dx)$. If $\mu \in ID$ has drift, then μ satisfies (5.8), since $\varepsilon^{-1} \int_{|x| \leq \varepsilon} |x|^2 \nu_\mu(dx) \leq \int_{|x| \leq \varepsilon} |x| \nu_\mu(dx)$.

2. Let Λ_h be one of $\bar{\Phi}_{p,1}$ with $p \geq 1$, $\Lambda_{q,1}$ with $q \geq 1$, and $\Psi_{1,\beta}$ with $\beta > 0$. Then any $\mu \in \mathfrak{R}^e(\Lambda_h)$ satisfies (5.1) and any $\mu \in \mathfrak{R}^e(\Lambda_h^*)$ satisfies (5.8). To see this, first note that if $\mu \in \mathfrak{R}^e(\Lambda_h)$ [resp. $\mathfrak{R}^e(\Lambda_h^*)$] and if $\mu = \mu_0 * \mu_1$ with $\mu_0 \in ID_0$ and μ_1 being Gaussian, then $\mu_0 \in \mathfrak{R}^e(\Lambda_h)_0$ [resp. $\mathfrak{R}^e(\Lambda_h^*)_0$]; see Proposition 3.18 of [9]. Then, for $\mathfrak{R}^e(\Lambda_h)$, note that any function monotone of order $p \geq 1$ is decreasing to 0 (Corollary 2.6 of [9]) and use Lemma 4.2 of [5]. For $\mathfrak{R}^e(\Lambda_h^*)$, use the following analogue of Lemma 4.2 of [5]: *Under the assumption that $\mu \in ID$ is such that ν_μ has a rad. dec. $(\lambda(d\xi), u^{-2}l_\xi(u)du)$ with $l_\xi(u)$ measurable in (ξ, u) and increasing in $u \in \mathbb{R}_+^\circ$, we have $l_\xi(0+) = 0$ for λ -a. e. ξ if and only if (5.8) holds.* This is because

$$\varepsilon^{-1} \int_{|x| \leq \varepsilon} |x|^2 \nu_\mu(dx) = \int_S \lambda(d\xi) \int_{(0,1]} v^2 v^{-2} l_\xi(\varepsilon v) dv.$$

Now let $\mu \in \mathfrak{R}^e(\Lambda_h^*)$. In order to prove that μ satisfies (5.8), let us show that ν_μ has a rad. dec. $(\lambda(d\xi), u^{-2}l_\xi(u)du)$ with $l_\xi(u)$ measurable in (ξ, u) , increasing in $u \in \mathbb{R}_+^\circ$, and $l_\xi(0+) = 0$ for λ -a. e. ξ . Indeed, if $\Lambda_h = \bar{\Phi}_{p,1}$ with $p \geq 1$, then ν_μ has a rad. dec. $(\lambda(d\xi), u^{-p-1}k_\xi(u)du)$ with $k_\xi(u)$ increasing of order p on \mathbb{R}_+° by Theorem 4.4 of [11] and $l_\xi(u) = u^{1-p}k_\xi(u)$ is increasing in u and $l_\xi(0+) = 0$, since $u^{p-1}k_\xi(u^{-1})$ is monotone of order p in $u \in \mathbb{R}_+^\circ$ by Proposition 4.3 of [11]. If $\Lambda_h = \Lambda_{q,1}$ with $q \geq 1$, then ν_μ has a rad. dec. $(\lambda(d\xi), u^{-2}h_\xi(\log u)du)$ with $h_\xi(y)$ being increasing of order q in $y \in \mathbb{R}$ by Theorem 4.5 of [11] and hence $h_\xi(y)$ is increasing and tends to 0 as $y \rightarrow -\infty$ by Proposition 4.3 of [11]. If $\Lambda_h = \Psi_{1,\beta}$ with $\beta > 0$, then ν_μ has a rad. dec. $(\lambda(d\xi), u^{-2}k_\xi(u^{-\beta})du)$ with $k_\xi(v)$ completely monotone in $v \in \mathbb{R}_+^\circ$ by Theorem 4.6 of [11] and thus $l_\xi(u) = k_\xi(u^{-\beta})$ is increasing in $u \in \mathbb{R}_+^\circ$ and $l_\xi(0+) = 0$.

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